GAUSS DIAGRAMS OF 3-MANIFOLDS

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ABSTRACT. The paper presents a simple combinatorial method to encode 3-dimensional manifolds, basing on their Heegaard diagrams. The notion of a Gauss diagram of a 3-manifold is introduced. We check the conditions for a Gauss diagram to represent a closed manifold and a manifold with boundary.

1. Introduction

Orientable 3-manifolds can be easily described by their Heegaard splittings. The manifold is presented as a union of two handlebodies, glued along the boundary by a homeomorphism of a splitting surface to itself. Any 3-manifold possesses a Heegaard splitting, though not a unique one. However, one can pass from one Heegaard splitting to another by a sequence of some standard simple operations. See, e.g., [FM]. A standard simple way to describe a Heegaard splitting of a 3-manifold is by its Heegaard diagram. A Heegaard diagram consists of two families of disjoint simple closed curves on a splitting surface. The families intersect, producing an immersed family of curves on the surface.

There is a simple combinatorial way to encode immersed curves in a plane, namely a Gauss diagram. The curve is presented as a circle in the plain with double points joined with a chord. One can also encode the information on the overpasses and underpasses, changing these diagrams into diagrams for knots and links. This presentation of plane curves and knot diagrams proved useful for computational purposes, especially for finite type knot and link invariants. The same construction can be used to describe immersed curves on a surface.

A manifold is described by curves on the surface and the curves are described by Gauss diagrams. It seems natural to introduce a new notion of Gauss diagrams of 3-manifolds, which appears to be a simple combinatorial way to describe 3-manifolds. In Section 2 we introduce the notion of a Gauss diagram corresponding to a Heegaard diagram

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of a closed orientable manifold. We show that the splitting surface and the entire manifold can be reconstructed from it. The equivalence of Gauss diagrams is defined and we prove that a closed orientable 3-manifold is determined by the equivalence class of its Gauss diagram. Next we show how to compute the fundamental group and the first homology group of a manifold directly from a Gauss diagram, and how to distinguish homology spheres.

A picture looking like a Gauss diagram does not necessary come from a closed orientable manifold. In Section 3 we consider all diagrams looking like a Gauss diagram, calling them abstract Gauss diagrams. We show that this construction allows to describe oriented manifolds with boundary. Later we formulate the conditions for a Gauss diagram to represent a closed orientable manifold and a complement of a knot in some closed orientable manifold.

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2. From 3-manifolds to Gauss diagrams and back again

2.1. **Heegaard and Gauss diagrams.** Let M be a closed orientable 3-manifold. A Heegaard splitting $(H^{\pm}, \Sigma)$ of M is a union of two handlebodies $H^{\pm}$, glued along their boundary Σ by a homeomorphism. The genus g of Σ is called the genus of the Heegaard splitting. One of $H^{\pm}$ can be always taken to be standard. The main property of a handlebody H is that there exists a set of discs $\Delta \subset H$, called a meridional disc system of H, such that $H - \Delta$ is a union of 3-balls. One can always find a minimal set of this sort, which means that $H - \Delta$ is a single ball. Since there is a unique way to fill in the ball, the handlebody can be easily reconstructed, if a meridional disc system is specified. This means that it is sufficient to specify the boundaries of the meridional discs on the surface. Let $\Delta^{\pm}$ be minimal meridional disc systems of $H^{\pm}$ respectively. The boundaries of these discs are two families $M^{\pm}$ of disjoint simple closed curves on Σ, from which the Heegaard splitting can be reconstructed by gluing 2-handles along the curves of $M^{\pm}$ and filling in 3-balls. The pair $(\Sigma, M^{\pm})$ is called a *Heegaard diagram* of M.

We think of both families as an immersed family of curves on the surface. In order to describe them in a more simple, combinatorial way, one can use Gauss diagrams, in a way similar to the plane curves. A *Gauss diagram* is the immersing collection of circles with the preimages

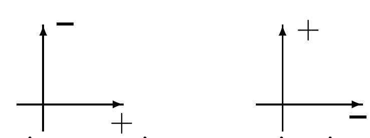
of each double point connected with a chord. Each chord is given a sign depending on whether the frame of tangents to the branches of the curve in this point coincides with the orientation of the plane. One can also associate each circle a Gauss word by traveling once around the circle and recording the chords with signs.

Gauss words were introduced by Gauss in [G], and the planarity problem was extensively studied. See, e.g. [LM], [RR], [FrM]. J.S. Carter proved in [C] that if two sets of immersed curves fill the surface in the sense that the complement regions are discs, then they are stable geotopic if and only if they have equivalent Gauss paragraphs. The notion of Gauss diagrams for knots and links was introduced by Polyak and Viro in [PV]. They used Gauss diagrams enhanced with the information on the overpasses and underpasses. The equivalence classes of abstract Gauss diagrams were called virtual knots by Kauffman in [K], since only part of the diagrams were realizable.

2.2. **From Heegaard diagram to Gauss diagram.**

Definition 2.1. *A* Gauss diagram $G = (\mathcal{M}^+, \mathcal{M}^-, h, \varepsilon)$ *consists of two families* $\mathcal{M}^\pm$ *of* g *disjoint oriented circles* $\mu_1^\pm, \dots, \mu_g^\pm$*, a family* $h = (h_1, \dots, h_n)$ *of chords joining the circles of* $\mathcal{M}^+$ *to those of* $\mathcal{M}^-$*, and the signs* $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ *of the chords respectively.*

Let $(\Sigma, M^\pm)$ be a Heegaard diagram of a closed orientable 3-manifold M. We associate to it a Gauss diagram $G(\Sigma, M^\pm)$ in a following way: each closed curve $m_i^\pm \in M^\pm$ can be regarded as an image of an immersion $f^\pm \colon \bigsqcup S^1 \to \Sigma$. Let $\mathcal{M}^\pm$ be two families of the preimages of $M^\pm$. The curves of different families intersect, and the intersection points have two preimages, one in each family. We join these points with a chord, joining a circle in $\mathcal{M}^+$ with a circle in $\mathcal{M}^-$. No chords join the circles of the same family. Each chord is given a sign of the intersection, depending on whether the orientation given by the frame (tangent to m_i^+ in p, tangent to m_j^- in p) coincides with the orientation of Σ, see Figure 2.1. This construction depends on the choice of the orientation of the curves of $M^\pm$.



positive intersection negative intersection

Figure 2.1. Signs of an intersection

The choice of orientation of curves motivates the following notion of the ε-equivalence:

Definition 2.2. *An ε-move is reversing the orientation of a circle together with reversing the signs of all chords ending on this circle. Two Gauss diagrams G_1 and G_2 are ε-*equivalent *if one can be obtained from the other by a sequence of ε-moves.*

A Gauss diagram can be viewed as a picture with two rows of circles joined with chords. For convenience, we assume the following conventions, when drawing a Gauss diagram: the circles of $\mathcal{M}^+$ are placed in the top row and are oriented counterclockwise, while the circles of $\mathcal{M}^-$, placed in the bottom row, are oriented clockwise.

2.3. **From Gauss diagrams to ribbon graphs.** Given a Gauss diagram associated to some closed orientable 3-manifold, one might think that only the information on the intersections is available and the information on the surface is lost. It turns out, however, that this information is enough to reconstruct the entire manifold. See Theorem 2.16. This result is based on a construction of a surface, associated to a Gauss diagram. First we construct a ribbon graph from a Gauss diagram, then glue discs with holes to the boundary components of this graph.

Let G be a Gauss diagram. Contract all chords to points, obtaining an oriented graph Γ. Each vertex and each edge of this graph is equipped with a sign. The signs of the vertices are the signs of the chords they come from, where the signs of the edges signify the family $\mathcal{M}^\pm$ the edge belongs to. This graph will be the core of the ribbon graph.

Consider two families of g oriented annuli as a product of $\mathcal{M}^\pm$ with $I = [-1, 1]$. Let $p^\pm$ denote the ends of a chord h in the circles $\mu^\pm$ in $\mathcal{M}^\pm$ respectively. Let $U_p^\pm \subset \mu^\pm$ be small neighborhoods of $p^\pm$, their direction induced by the orientation of $\mu^\pm$. We identify the square $I \times I$ with $U_p^+ \times I$ by a homeomorphism ϕ^+ such that $\phi^+(U_p^+ \times \{0\})$ coincides with the positive x direction in $I \times I$ and the orientation is preserved. We identify (by an orientation-preserving homeomorphism ϕ^-) $I \times I$ with $I \times U_p^-$ such that $\phi^-(\{0\} \times U_p^-)$ coincides with the positive y direction for a positive chord and with negative y direction for a negative chord. A homeomorphism $\phi^+ \circ (\phi^-)^{-1}$ identifies the small squares in these annuli at the neighborhood of each $p^\pm$ cross-like, as in Figure 2.2.

Using such a homeomorphism for all chords, one obtains a ribbon graph $\Gamma(G)$ with core Γ. We will call it *a ribbon graph associated to the*

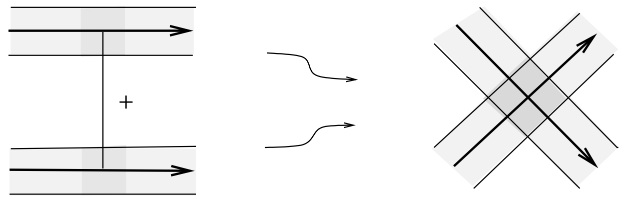


FIGURE 2.2. Gluing the bands

Gauss diagram. A ribbon graph can be viewed as a union of ribbons with bases glued to small squares, called coupons by Reshetikhin and Turaev in [RT]. We will treat ribbon graphs in a similar way.

Proposition 2.3. *Let $(\Sigma, M^{\pm})$ be a Heegaard diagram of some closed orientable 3-manifold M, and let N be a regular neighborhood of the union $M^+ \cup M^-$ in Σ. Let $G(\Sigma, M^{\pm})$ be a Gauss diagram associated to $(\Sigma, M^{\pm})$. Then $(\Gamma(G), \Gamma)$ and $(N, M^+ \cup M^-)$ are homeomorphic as pairs.*

Proof. We have two ribbon graphs N and $\Gamma(G)$. By construction of $\Gamma(G)$ we have the same number of coupons and ribbons in both graphs. One can identify all the coupons according to the order of intersections in one of the families $M^{\pm}$, say M^+. When identifying the coupons, one has to take care of the identification of the ribbon bases. However, the Gauss diagram provides the exact order to do it, since the cyclic order of the edges joining the vertex is known. Since the order of the identification of the coupons is given by the order of edges in M^+, meaning that we come to the coupon by a specific edge, the identification of the ribbon bases is unique. Since both ribbon graphs are oriented as surfaces and the identification goes along annuli, this can be done in orientation-preserving way. Thus the bases of the ribbons corresponding to the edges of M^- are identified. The cores of these ribbons are identified by the obvious homeomorphism, hence these ribbons can be also identified now, in the orientation-preserving way. □

2.4. **From ribbon graphs to splitting surfaces.** A ribbon graph $\Gamma(G)$ is an oriented surface with boundary. As Reshetikhin and Turaev mention in [RT], we can think that the bands of an oriented ribbon graph have two sides and we always see only one of them. We assume that this side is such that the boundary arc whose direction coincides with the orientation of the edge is always seen on the right side of the edge, as in Figure 2.5.

If our Gauss diagram is associated to some Heegaard diagram $(\Sigma, M^{\pm})$ of a closed orientable manifold, this graph is a ribbon graph on the surface Σ. Each family $M^{\pm}$ separates Σ into a sphere with holes, hence the complement of this graph in the surface is a collection of discs with holes. We would like to specify the sequences of edges the boundaries of these discs are glued along. We follow the construction of J.S. Carter in [C] for Gauss paragraphs, adapting his construction to the case of Gauss diagrams.

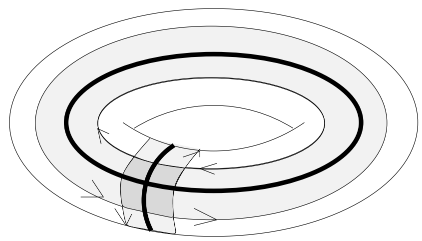


FIGURE 2.3. A ribbon graph on a surface

If one is approaching the vertex, while going along a boundary component, he has to turn right - see Figure 2.3. In terms of Gauss diagrams, a turn in a vertex of Γ means that if one approaches a chord, going along a curve of the top family $\mathcal{M}^+$, he has to descend using a chord and continue along the curve of $\mathcal{M}^-$, and vice versa. This rule can be applied to any Gauss diagram, not necessary to the one associated to some Heegaard diagram of some manifold. The "right turn rule" then means that for the positive chord one has to cross it on ascending path and to continue along it on the descending path. See Figure 2.4a. Approaching the negative chord, one ascends along the chord and crosses it on his way downwards. See Figure 2.4b.

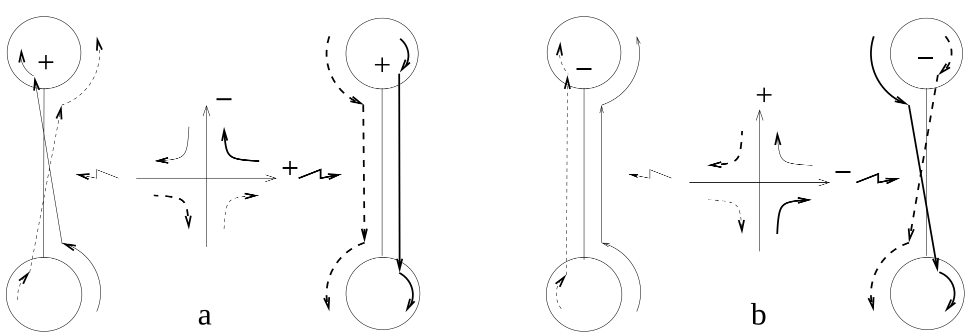


FIGURE 2.4. Right turn rule on positive and negative intersections

A sequence of edges corresponding to the boundary component is called a *cycle* in a Gauss diagram. We are going to glue discs with holes to boundary components.

Each edge of the Gauss diagram G produces two arcs in the boundary of $\Gamma(G)$. We would like to know directly from G which boundary component contains each of these arcs. It proves useful for computational purposes to include in the Gauss diagram the information which pair of cycles the edge belongs to. An easy way to encode it is using the elements of a finite set C. Elements of C are in one-to-one correspondence with boundary components of $\Gamma(G)$. We associate to each edge of G an ordered pair $(c_1, c_2), c_i \in C$. The first element corresponds to the boundary component having the same direction as the edge itself and the second one - to the opposite one.

In these terms the "right turn rule" will look as follows. Let (a_1, a_2), (b_1, b_2) be the pairs assigned to incoming and outgoing edges of μ^+ for some chord p, $(c_1, c_2), (d_1, d_2)$ - the pairs assigned to incoming and outgoing edges of μ^- respectively. See Figure 2.5.

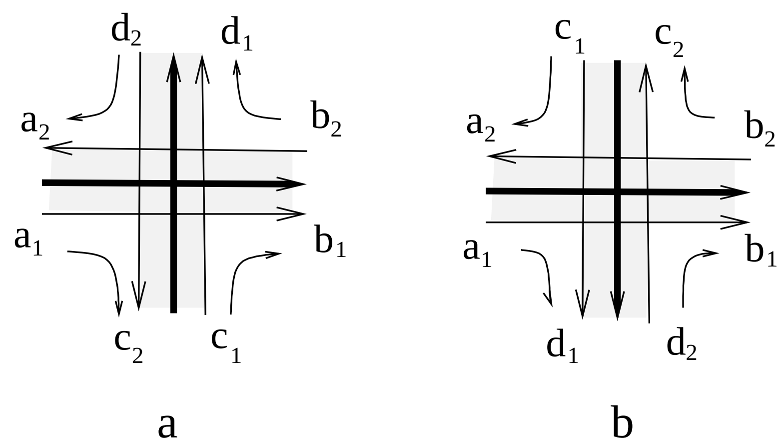


FIGURE 2.5. Right turn rule in terms of colors

For positive intersection we have: $(a_1, a_2, b_1, b_2) = (c_2, d_2, c_1, d_1)$, see Figure 2.5a, and for negative one we have: $(a_1, a_2, b_1, b_2) = (d_1, c_1, d_2, c_2)$, see Figure 2.5b. From now on we incorporate this information into Gauss diagram, without changing the notation.

We would like to construct a closed surface from $\Gamma(G)$, hence we glue discs with holes to it. The result of this construction is not unique. If $G = G(\Sigma, M^{\pm})$, we would like to obtain the surface homeomorphic to Σ and to get a Heegaard diagram producing the same manifold. To do this one has to know which boundary components will be glued to the

same disc with holes. This leads us to the notion of a decorated Gauss diagram.

Consider an equivalence relation on the set C of all the cycles of $G(\Sigma, M^{\pm})$, where two cycles will be equivalent if they correspond to the boundary components of the same disc with holes in $\Sigma - N$. It is convenient to define this equivalence relation as a coloring of the elements of C by the colors $c_1, \dots, c_m$ of the corresponding discs with holes.

A *decorated Gauss diagram* $G^d = (\mathcal{M}^+, \mathcal{M}^-, h, \varepsilon, c)$ is a Gauss diagram with a color c_i associated to each cycle, where the colors are members of a finite set $c = (c_1, \dots, c_m)$. We associate to a Heegaard diagram $(\Sigma, M^{\pm})$ a decorated Gauss diagram $G^d(\Sigma, M^{\pm})$. The only thing to do is to define the coloring of the cycles. All boundary components of the same connected component of $\Sigma - N$ have the same color. To construct the surface associated to G^d, for each color pick a sphere with a number of holes equal to the number of cycles having this color and glue it to all these cycles. This surface will be denoted by $S(G^d)$ and will be called *the surface associated to the decorated Gauss diagram*. The manifold associated to it can be constructed by gluing discs to $\mathcal{M}^{\pm}$ and capping the spheres with 3-balls.

Proposition 2.4. *Let $(\Sigma, M^{\pm})$ be a Heegaard diagram of a closed orientable 3-manifold M. The surface $S(G^d)$ associated to $G^d(\Sigma, M^{\pm})$ is homeomorphic to Σ and the associated manifold is homeomorphic to M.*

Proof. Since each of the families $M^{\pm}$ separates Σ into discs with holes, the surface $\Sigma - N$ consists of discs with holes, too. By Proposition 2.3, we have a homeomorphism $\varphi\colon \Gamma(G^d) \to N$. Consider all the cycles of $G^d(\Sigma, M^{\pm})$ colored by the same color c_i. All of them correspond to the boundary components of $\Gamma(G^d)$, whose images under φ are glued to the same disc with holes in $\Sigma - N$. The discs with holes glued to the boundary components of $\Gamma(G^d)$ and the components of $\Sigma - N$ are homeomorphic. The homeomorphism is defined by the image of φ on the boundaries.

Since the Heegaard diagram was reconstructed, the manifold obtained by gluing discs along the core of $\Gamma(G^d)$ will be homeomorphic to M. □

Remark: If $\Sigma - N$ is a union of discs, then all cycles of $G^d(\Sigma, M^{\pm})$ have different colors. This means that a manifold can be reconstructed from a non-decorated Gauss diagram, and we can omit the decorations.

2.5. **From decorated Gauss diagrams back to non-decorated ones.** Suppose that on the surface Σ there are two curves belonging to different families $M^{\pm}$, which can be isotoped along Σ, such that a pair of cancelling intersections appears, in a manner similar to the second Reidemeister move in knots. See Figure 2.6. In a Gauss diagram this will look like adding a pair of adjacent chords with opposite signs.

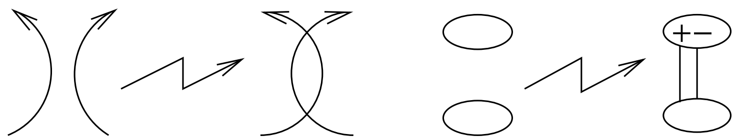


FIGURE 2.6. A pair of cancelling intersections

Consider a decorated Gauss diagram G^d. Let $e^+ \in \mathcal{M}^+, e^- \in \mathcal{M}^-$ be two edges, belonging to two cycles of the same color. Let $(a_1, a_2), (b_1, b_2)$ be the respective colorings of e^+, e^-, and let p^+, m^+ be points of e^+, p^-, m^- be points of e^-. The chord joining p^+ with p^- will be positive and the chord joining m^+ with m^- will be negative. Going along $e^{\pm}$ induces an order on the points of an edge. We write $p < m$ if p appears before m. The signs and the order of the chords in $e^{\pm}$ depends on the colors of the edges:

- if $a_1 = b_1$ then $m^+ < p^+$ and $p^- < m^-$
- if $a_1 = b_2$ then $p^+ < m^+$ and $p^- < m^-$
- if $a_2 = b_1$ then $m^+ < p^+$ and $m^- < p^-$
- if $a_2 = b_2$ then $p^+ < m^+$ and $m^- < p^-$.

This rule explains which sides of the bands meet when two chords appear while two cancelling intersections are created. See Figure 2.7 for an example of two bands meeting with $a_2 = b_2$.

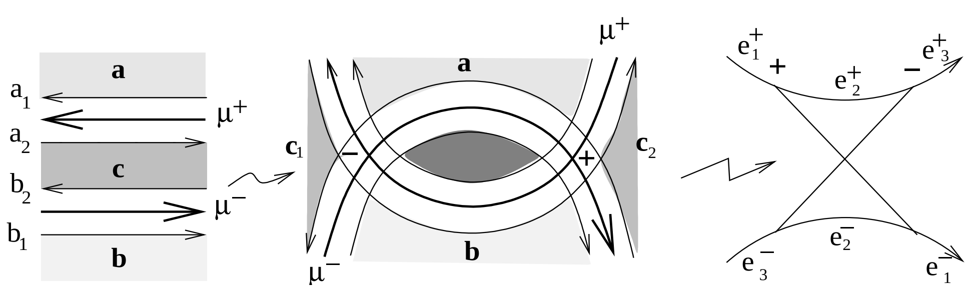


FIGURE 2.7. Two bands meeting with $a_2 = b_2$.

We will call this change in a decorated Gauss diagram an *R-move*.

During an R-move three new edges $e_1^\pm, e_2^\pm, e_3^\pm$ are created instead of each of $e^\pm$, so that $e_1^\pm$ have a common chord, and $e_3^\pm$ too. See Figure 2.7. The colorings of these edges are given by the following.

If $e^\pm$ belonged to the same cycle of color c, then there is a partition of the set of all cycles colored by c into two sets, colored into two new colors that did not exist in the color set of G. Two new cycles, containing $e_1^\pm$ and $e_3^\pm$ respectively, belong to different sets. A new cycle, consisting of two edges e_2^+, e_2^- has an entirely new color d.

If $e^\pm$ belonged to two different cycles of color c, then two new cycles appear. One of them contains all the edges of the old cycles and two pairs $e_1^\pm, e_3^\pm$ and has the same color c. The other one consists only of e_2^+, e_2^- and has a new color d.

The edges $e_1^\pm$ and $e_3^\pm$ have the same coloring pairs as $e^\pm$, save that c can be replaced by a new color, as mentioned above. The edges $e_2^\pm$ have a new coloring, obtained from the coloring of $e^\pm$ by the following changes: c is replaced by d, the second color in the pairs is switched between e^+ and e^-, and the colors inside each pair are also switched. E.g., if e^+ is colored by (a, c) and e^- - by (b, c), as in Figure 2.7, the edge e_2^+ is colored by (d, b) and e_2^- - by (d, a).

The opposite move to an R-move is called R^{-1}-*move.*

Definition 2.5. *Two decorated Gauss diagrams are called* R-equivalent *if one can be obtained from the other by a sequence of* R *and* R^{-1}*-moves.*

Remark: R-equivalent diagrams have the same Heegaard splitting.

Remark: An R-move reminds us the second Reidemeister move in knots. There is no equivalent to the third Reidemeister move, since three curves intersecting would produce an intersection between the curves of the same family, which is forbidden.

A ribbon graph $\Gamma(G^d)$ may be connected, and may be not. This leads us to the notion of R-connected diagram. Let G^d be an abstract decorated Gauss diagram. Consider a graph, whose vertices are connected components of G^d. Two vertices are connected with an edge if and only if the sets of colors appearing on these connected components have a color in common. We call the diagram G^d *R-connected* if and only if this graph is connected.

Lemma 2.6. *The surface* $S(G^d)$ *is connected if and only if* G^d *is* R-*connected.*

Proof. The connected components of $\Gamma(G^d)$ are exactly the connected components of G^d itself. Two components will have the same disc

with holes glued to both of them if and only if they have a color in common. Hence the connected components of the graph defined above correspond to connected components of $S(G^d)$. $\square$

Proposition 2.7. *Any decorated Gauss diagram G^d is R-equivalent to a Gauss diagram with the number of colors used on it equal to the number of cycles. The resulting diagram is connected if and only if G^d is R-connected.*

Proof. Consider a decorated Gauss diagram G^d. Let Σ_i be a non-simply connected component of $S(G^d)-\Gamma(G^d)$. It is a disc with holes and each of its boundary components contains arcs of both families. Perform an R-move between two different boundary components of Σ_i. Such a move produces two components, such that one of them is a disc with a new color. The other one has one hole less than the initial, thus lowering the number of cycles colored by the same color. See Figure 2.8. An R-move does not change the number of boundary components of any other component of $S(G^d) - \Gamma(G^d)$. The resulting diagram will be R-equivalent to the previous one. One can continue this process till Σ_i turns into a disc.

Remove all non-simply-connected components of $S(G^d) - \Gamma(G^d)$ by the R-moves. This will give a decorated Gauss diagram with all cycles colored differently.

If the Gauss diagram $G^d(\Sigma, M^{\pm})$ is not connected, we assume for convenience that it has two connected components. If G^d is R-connected, there are two cycles colored by the same color in different components of $G^d(\Sigma, M^{\pm})$. An R-move between these cycles will change the diagram into a connected one. $\square$

Corollary 2.8. *For any Heegaard diagram $(\Sigma, M^{\pm})$ of a closed connected orientable 3-manifold there is a connected Gauss diagram G with all cycles colored in different colors R-equivalent to $G^d(\Sigma, M^{\pm})$. Moreover, there is a Heegaard diagram isotopic to $(\Sigma, M^{\pm})$, such that G is associated to it.*

2.6. **Gauss diagrams and their associated surfaces.** To compute the genus of the surface associated to an abstract Gauss diagram, we will use the notion of the *excess of a coloring*, which is the difference Δ_c between the number of cycles and number of colors. Obviously, if all cycles have different colors, Δ_c vanishes.

Proposition 2.9. *The genus of a surface $S(G^d)$ associated to an abstract decorated Gauss diagram G^d is $g_{S(G^d)} = 1 + \Delta_c + \frac{1}{2}(|h| - |c|)$, where $|c|$ is the number of cycles and $|h|$ is the number of chords in G^d.*

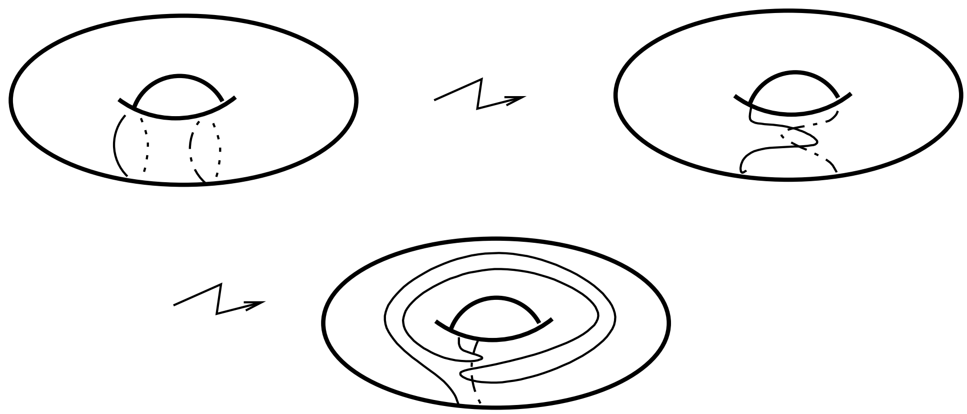


FIGURE 2.8. Adding pairs of cancelling intersections changing annuli into discs

Proof. This can be done by computing the Euler characteristic of $S(G^d)$. The surface $S(G^d)$ is constructed by gluing discs with holes to the boundary components of a ribbon graph $\Gamma(G^d)$. The number of vertices of $\Gamma(G^d)$ is the number of chords $|h|$. The number of edges of $\Gamma(G^d)$ equals twice the number of chords. We do not count circles that have no chord attached to them, since they produce annuli in $\Gamma(G^d)$, and their Euler characteristic is 0. Thus $\chi(\Gamma(G^d)) = |h| - 2|h| = -|h|$. Glue discs with holes to $\Gamma(G^d)$. For each of them, the number of the boundary components equals the number $|c_i|$ of cycles colored by c_i. The total number of these discs with holes is the number of colors $|c| - \Delta_c$. The Euler characteristic of each such disc with holes is $\chi_i = 2 - |c_i|$. Summarizing, we have:

$$\chi(S(G^d)) = -|h| + \sum_{i=1}^{|c|-\Delta_c} \chi_i = -|h| + 2|c| - 2\Delta_c + \sum_{i=1}^{|c|-\Delta_c} |c_i|.$$

The sum over $|c_i|$ equals the number of cycles $|c|$, giving $\chi(S(G^d)) = -|h| + |c| - 2\Delta_c$. The genus is given by $\chi(S(G^d)) = 2 - 2g_{S(G^d)}$. □

A surface associated to a non-decorated Gauss diagram G is a surface obtained by gluing discs to all boundary components of $\Gamma(G)$. In other words, this is the surface associated to a decorated Gauss diagram with all cycles colored in different colors.

Corollary 2.10. *The genus of the surface associated to the connected non-decorated Gauss diagram is* $g_{S(G)} = 1 + \frac{1}{2}(|h| - |c|)$.

Proposition 2.11. *Let* $(\Sigma, M^{\pm})$ *be a Heegaard diagram of some closed orientable 3-manifold* M. *Assume that* $G(\Sigma, M^{\pm})$ *is connected. Let*

g be the number of curves in each family $M^{\pm}$. If $g_{S(G)} = g$, then $\Sigma - (M^{+} \cup M^{-})$ is a collection of discs, otherwise $(\Sigma, M^{\pm})$ is reducible.

Proof. If $g_{S(G)} = g$, the surface associated to a non-decorated Gauss diagram is homeomorphic to Σ. The boundaries of the meridional disc system are also reconstructed, hence Σ is separated into discs by the curves of $(M^{+} \cup M^{-})$.

If $g_{S(G)} \neq g$, then cutting $S(G)$ along the curves of the diagram produced surfaces other than discs. Thus any essential closed curve in such a non-simply connected component bounds a disc in both handlebodies of the Heegaard splitting. This means that the Heegaard splitting is reducible. □

2.7. **Equivalent diagrams.** We introduce now two types of moves of Gauss diagram motivated by some equivalences of Heegaard splittings.

Suppose that two curves m_1, m_2 on the Heegaard diagram $(\Sigma, M^{\pm})$, belonging to the same family $M^{\pm}$, say M^{+}, can be joined together with a simple curve α disjoint from all other curves of $M^{\pm}$. Let α', α'' be the boundary components of a thin collar $N(\alpha) = \alpha \times [-1, 1]$ of α in $\Sigma - (M^{+} \cup M^{-})$. Set

$$m_2' = (m_2 - N(\alpha)) \cup (m_1 - N(\alpha)) \cup \alpha' \cup \alpha''.$$

By a slight push make this curve disjoint from m_1. This does not change the handlebodies of the splittings, only the meridional discs defining them. The Heegaard diagram $(\Sigma, (M^{+} - m_2) \cup m_2', M^{-})$ is obtained from the diagram $(\Sigma, M^{\pm})$ by a handle slide of m_2 along m_1. See Figure 2.9.

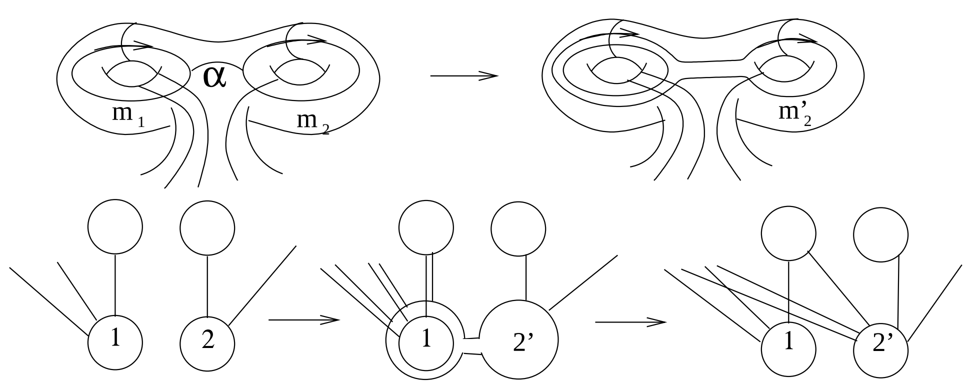


FIGURE 2.9. Handle slide

This process changes the Gauss diagram of M, with a small section of μ_2 being substituted by a curve equal to μ_1 without a small segment. All the curves that intersected m_1 now intersect m_2' also, which means

that we have the copies of all chords that joined μ_1 now joining μ_2'. The signs of the intersections (and the chords) are the same, if the direction of the segment removed coincides with the direction of the added segment. If not, one has to change the direction of the added segment, and reverse the signs for all chords joining it, as in the case of ε-equivalence. Clearly, the move is allowed only if the segments removed from both cycles are of the same color. We say that the resulting diagram is obtained from the initial one by an *H-move*. The opposite of this move will be called *H^{-1}-move*.

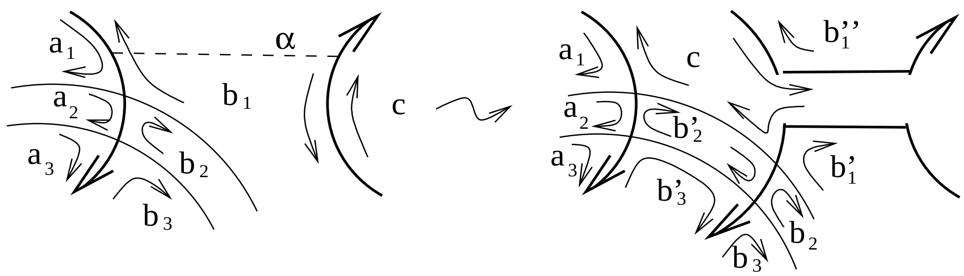


FIGURE 2.10. Color change during the handle slide

The change of the colors in the case of handle slide is illustrated in Figure 2.10. The color of the region is the color of the cycle bounding it. Let (a_i, b_i) be the colorings of the edges e_i of μ_1, and let α join e_1 and $e \in \mu_2$. Let (b_1, c) be the coloring of e. The edges e_i, save e_1, will now be parts of new small cycles, consisting only of 4 edges each and having new colors b_i'. The edges $e_i, i > 1$ will have colors (a_i, b_i'). The edge e_1 will have the coloring (a_1, c), if the directions of the added and removed segments coincide. In this case also the edges e_i' added to μ_2 have colors (b_i', b_i) and belong to the same new cycles. The edges e_i' have colors (b_i, d_i). The remaining two edges of these new cycles have the colors (b_i', b_{i+1}') or (b_{i+1}', b_i'), depending on the signs of their ends. We described already the colors of all e_i' save the first and the last ones. Their coloring equals that of e if α joined two edges of different cycles. If not, the color b_1 splits and the coloring is as in Figure 2.10. If the direction of the added segment is opposite to that of the removed one, the colorings of the edges of the added segment switch.

Lemma 2.12. *H-move does not change the genus of $S(G^d)$.*

Proof. Consider the slide of μ_i along μ_j. Let $|h_j|$ be the number of chords joining μ_j. Then the total number of chords becomes $|h| + |h_j|$. The number of cycles and colors depend on whether the edges joined by α belonged to the same cycle or not. If α connected two edges

of the same cycle, two cycles are created instead of it. Their colors are different, and the set of the cycles colored by the initial color is partitioned into two sets, each with a new color. Additional $(|h_j|-1)$ new 4-edged cycles are also created, each with a separate color. Hence both the number of colors and the number of cycles increases by $|h_j|$, and Δ_c does not change. The formula of Proposition 2.9 for the changed diagram G'^d gives: $g_{S(G'^d)} = 1+\Delta_c+\frac{1}{2}(|h|+|h_j|-|c|-|h_j|) = g_{S(G^d)}$. See also Figure 2.10.

If α connected two edges of two different cycles of the same color, these cycles are now replaced by a single one. Hence the number of colors increases only by $(|h_j|-1)$, while the number of cycles increases by $(|h_j|-2)$. The formula of the Proposition 2.9 then gives us: $g_{S(G'^d)} = 1+\Delta_c-1+\frac{1}{2}(|h|+|h_j|-|c|-|h_j|+2) = g_{S(G^d)}$. $\square$

Another basic operation with Heegaard splittings is stabilization. This operation consists of adding a pair of cancelling handles. See Figure 2.11. One can think of it as a connected sum of M with S^3, where the ball of connection is a small neighborhood of a point on the splitting surface Σ disjoint from the curves of $M^{\pm}$, and S^3 is thought as having a standard genus 1 splitting $T^{\pm}$. The boundaries of the meridional discs of the solid tori $T^{\pm}$ intersect in one point and are disjoint from $M^{\pm}$. This means that in terms of Gauss diagrams the stabilization can be viewed as adding a circle to each family $\mathcal{M}^{\pm}$. These two circles are joined with one chord (the sign depends on the orientation of the boundaries of the meridional discs of $T^{\pm}$ and can be chosen arbitrarily), and no chord joins them to the other circles. They produce a single cycle whose color is one of the existing colors. The color represents the place where the stabilization was made. We call this operation an *S-move*. There is, certainly, the inverse operation of removing two cancelling handles, disjoint from all others. This looks like removing a pair of circles joined by a single chord and not connected to any other circle of a Gauss diagram. We call it *S^{-1}-move*.

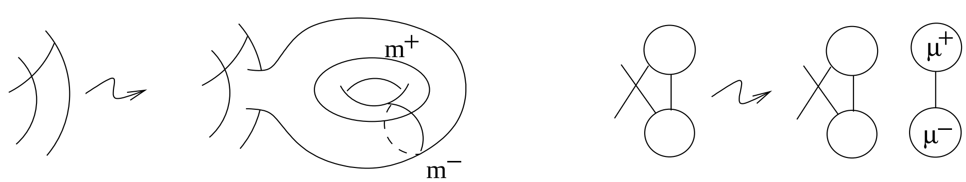


FIGURE 2.11. The stabilization on the Heegaard and Gauss diagrams

Lemma 2.13. *S-move increases the genus of $S(G^d)$ by 1.*

Proof. Let G'^d be a stabilization of G^d. A single cycle was added to G^d, and no new colors. Hence Δ_c increased by 1. We also have one chord more. Substituting to the formula of the Proposition 2.9 for $g_{S(G'^d)}$ we have: $g_{S(G'^d)} = 1 + \Delta_c + 1 - \frac{1}{2}(|h| + 1 - |c| - 1) = g_{S(G^d)} + 1$. $\square$

Definition 2.14. *Two Gauss diagrams are called HS*-equivalent *if one can be obtained from the other by a sequence of S-moves, H-moves and their opposites.*

2.8. **From splitting surfaces to 3-manifolds.** One of the problems of Heegaard splittings is that a 3-manifold can be presented by different Heegaard splittings (possibly of a different genus). Moreover, the same Heegaard splitting can be presented by different Heegaard diagrams. The *HS*-equivalence reflects this phenomenon on the level of Gauss diagrams.

Definition 2.15. *Two Gauss diagrams are* equivalent *if one can be obtained from the other by a sequence of ε, R, H, S-moves or their opposites.*

Theorem 2.16. *A closed connected orientable 3-manifold is in one-to-one correspondence with the equivalence class of its Gauss diagram.*

Proof. A well known theorem by Singer says that one can pass from one Heegaard splitting to any other Heegaard splitting of the same closed manifold by a consequent use of handle slides and stabilizations. See [S]. On the level of Gauss diagrams it means that the Gauss diagrams associated to these Heegaard splittings are *HS*-equivalent.

Now let $(\Sigma, M^{\pm})$ be a Heegaard diagram of some closed orientable 3-manifold M. By Corollary 2.8, there is a connected Gauss diagram G with all cycled colored differently, R-equivalent to $G(\Sigma, M^{\pm})$, and the curves of the Heegaard diagram can be isotoped so that G is the diagram associated to $(\Sigma, M^{\pm})$ after the changes. We will not change the notation for the isotoped $M^{\pm}$. Construct the surface associated to G. By Proposition 2.3, a ribbon graph associated to $G(\Sigma, M^{\pm})$ is homeomorphic to a regular neighborhood of $M^{\pm}$. Consider the boundary components of this neighborhood. Each of them is a simple closed curve disjoint from the curves of $M^{\pm}$, homotopic to the image of a cycle in $G(\Sigma, M^{\pm})$ under this homeomorphism, and all cycles of $G(\Sigma, M^{\pm})$ are presented by these boundary components. If this curve is essential in Σ, compress along it. Continue for all cycles. Take only those components of the resulting surface that contain curves of $M^{\pm}$. This surface is homeomorphic to the surface $S(G)$.

The complement of the ribbon graph in $S(G)$ is a collection of discs. In the surgered surface containing $M^{\pm}$, the complement of $M^{\pm}$ is also a

collection of discs, the boundary of each is a cycle in the Gauss diagram. A homeomorphism is constructed by taking the corresponding discs to each other and the correspondence is determined by the boundary. Thus any two Heegaard diagrams with equivalent Gauss diagrams can be obtained from a surface associated to their common Gauss diagram by a connected sum with unknotted tori and handle slides. □

2.9. **Visualizing Gauss diagrams.** The algorithm of Section 2.4 gives us the possibility to reconstruct the surface from the Gauss diagram, but it is hard to visualize it. The following algorithm allows us to reconstruct the picture of the Heegaard diagram in an easy and quite visual way, provided that a Gauss diagram comes from some Heegaard diagram of a closed orientable 3-manifold M and is connected. We also need to assume that $g = g_{S(G)}$, where $g_{S(G)}$ is given by the formula of Corollary 2.10.

Consider such a Gauss diagram G. Since one of the families $M^{\pm}$ of curves on the surface can be taken to be standard, we start with an oriented disc with $2g-1$ holes. Let the boundary have the orientation induced by the orientation of the disc. Split the set of the boundary components in pairs. Choose an orientation-reversing homeomorphism between the curves and identify them in pairs.

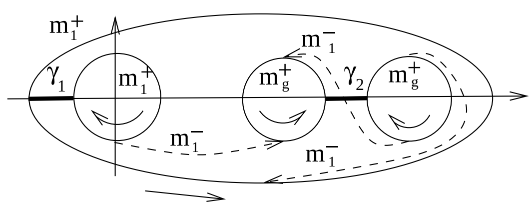


FIGURE 2.12. Surface S cut along the curves of M^+

This gives us a surface S of genus g. See Figure 2.12, where the boundary components already have reversed orientations, where necessary. After the identification we refer to the identified curves as $m_i^+,\ i = 1, \ldots, g$.

The curves m_i^+ can be considered as a map from a Gauss diagram to a Heegaard diagram, defining the closed curves belonging to one of the families. To get a map from $\mathcal{M}^-$ to S for all segments of the circles of $\mathcal{M}^-$ between the endpoints of the chords draw disjoint curves in S between the points corresponding to the upper ends of the chords intersecting this segment. When choosing the copy of m_i^+ that should contain the final point of the edge of m_j^-, one has to take the chord sign into account. For instance, taking the orientations of m_i's as in

Figure 2.12, m_1^- has to join the counterclockwise copy of m_2^+ for a negative intersection and the clockwise copy of m_2^+ for a positive one. The concatenation of these curves after identifying the m_i's will be a closed curve in S corresponding to a circle of $\mathcal{M}^-$.

The Gauss diagram associated to the Heegaard diagram obtained above is the same as G. By Theorem 2.16, the manifold given by this Heegaard diagram is homeomorphic to M.

2.10. **First simple computations using Gauss diagrams.** The fundamental group of a closed orientable 3-manifold M and its first homology group can be computed easily from a Gauss diagram G, associated to some Heegaard splitting of M. For each circle μ_j^- of $\mathcal{M}^-$ we write a word r_j in letters $g_i, i = 1, \dots, g$ in the following way: let $h_1^{\varepsilon_1}, \dots, h_n^{\varepsilon_n}$ be the sequence of chords with signs $\varepsilon_i \in \{\pm 1\}$ joining μ_j^- to the circles of $\mathcal{M}^+$. Suppose a chord $h_i^{\varepsilon_i}$ joins $\mu_j^- \in \mathcal{M}^-$ to $\mu_k^+ \in \mathcal{M}^+$. For such a chord we write $g_k^{\varepsilon_i}$ in r_j. Thus we get a cyclic word, which is a relation in the presentation of $\pi_1(M)$.

Proposition 2.17. *Let $(\Sigma, M^\pm)$ be a Heegaard diagram of some closed orientable 3-manifold M. Assume that $G = G(\Sigma, M^\pm)$ is connected and $g = g_{S(G)}$. Then $\pi_1(M)$ has a presentation $\langle g_1, \dots, g_g | r_1, \dots, r_g \rangle$, where g_i correspond to the circles of $\mathcal{M}^+$ and r_j correspond to the circles of $\mathcal{M}^-$ and are obtained as above.*

Proof. We use the algorithm and the notation of Section 2.9. Each g_i corresponds to a curve γ_i, see Figure 2.12. The fundamental group $\pi_1(M)$ can be obtained from the fundamental group of S by adding the relations given by the discs glued to the curves of the Heegaard diagram. The curves $m_i^+, \gamma_i,\ i = 1, \dots, g$ form the generator set for $\pi_1(S)$. When we glue discs to the curves corresponding to $\mathcal{M}^+$, we remain with γ_i's only. They will be the generators of $\pi_1(M)$. When one wants to count how many times does the curve wind along the handle defined by γ_i, he simply counts how many times it crosses some fixed curve dual to γ_i, which is m_i^+. This explains the form of the relations. □

Let a_{ij} denote the algebraic number of chords joining the circle μ_j^- in $\mathcal{M}^-$ to a circle μ_i^+ in $\mathcal{M}^+$.

Corollary 2.18. *The first homology group of M is the abelian group generated by $g_1, \dots, g_g$, where g_i correspond to the circles of $\mathcal{M}^+$. The relations are $r_1, \dots, r_g$, where $r_j = \sum_i a_{ij} g_i = 0$.*

The algebraic numbers of chords joining the circles can also be written in the intersection matrix A_{ij}. This matrix allows us to determine whether a manifold is a homology sphere.

2.11. **Examples.**

The 3-sphere: S^3 is the simplest example. We represent a Gauss diagram of the standard genus 1 splitting of it. See Figure 2.13.

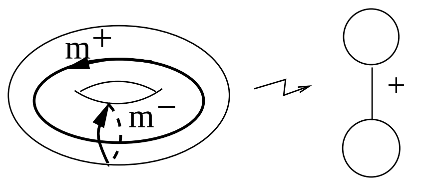


FIGURE 2.13. Heegaard and Gauss diagrams of S^3

Lens spaces: They are a bit more complicated. In Figure 2.14 we present the Gauss diagrams of $L(5,1)$ and $L(5,2)$. The signs of all chords are + on both diagrams. The fundamental group is the same, as one should expect:

$$\pi_1(L(5,q)) = \langle g | g^5 \rangle = Z_5, \ q = 1, 2.$$

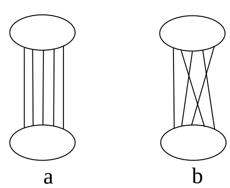


FIGURE 2.14. Gauss diagrams of $L(5,1)$(a) and $L(5,2)$(b)

Poincare homology sphere: Figure 2.15 represents a Heegaard diagram and the associated Gauss diagram of the Poincare manifold P^3, as it appears in [R, p.245]. The picture presented by Rolfsen looks exactly like the one obtained by the algorithm of Section 2.9 from a Gauss diagram. The fundamental group, according to the Proposition 2.17, is

$$\langle g_1, g_2 | g_1^{-4} g_2 g_1 g_2, g_1 g_2^{-2} g_1 g_2 \rangle = \langle g_1, g_2 | (g_1 g_2)^2 = g_1^5 = g_2^3 \rangle,$$

which is the binary icosahedral group, as expected. Abelinizing, we get $H_1(P^3) = \langle e \rangle$, hence P^3 is indeed homology sphere.

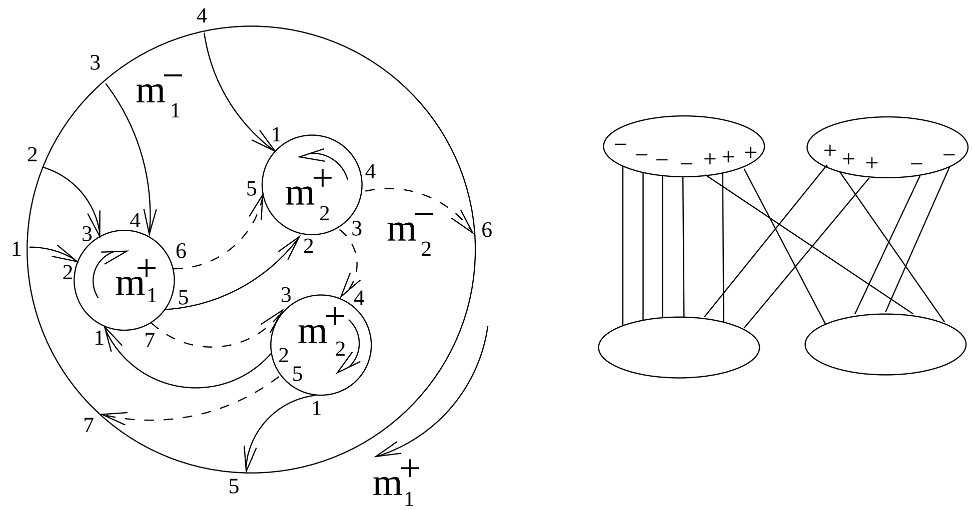


FIGURE 2.15. Heegaard and Gauss diagrams of P^3

Another homology sphere: We consider the example of a homology sphere given by Hempel in [He, p.19]. Its Heegaard diagram is shown in Figure 2.16, together with its Gauss diagram. One can easily calculate the fundamental group:

$$\pi_1(M) = \langle g_1, g_2 | g_1 g_2^{-1} g_1^{-1} g_2^2 g_1^{-1} g_2^{-1}, g_1 g_2 g_1 g_2^{-1} g_1^{-1} g_2^{-1} \rangle,$$

which gives trivial first homology group. The intersection matrix is in this case

$$\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

and its determinant is 1, in agreement with the expectations.

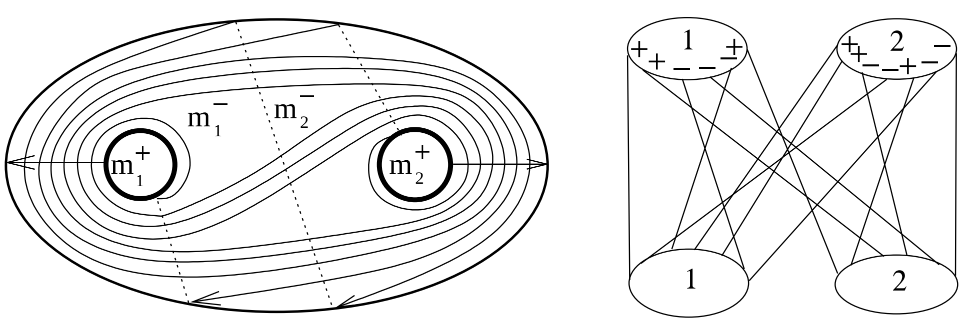


FIGURE 2.16. Heegaard and Gauss diagrams of a homology sphere from [He]

3. Abstract Gauss diagrams

A natural question that arises when dealing with Gauss diagrams of 3-manifolds is: given an abstract Gauss diagram, can one find a manifold that this Gauss diagram is associated to it?

Gauss diagrams for 3-manifolds closely resemble Gauss diagrams for knots and links. The equivalence classes of Gauss diagrams for links that do not necessarily correspond to a link are called virtual links. We can also call the equivalence classes of Gauss diagrams of 3-manifolds virtual manifolds.

Below we show that a 3-manifold, in general with a boundary, can be associated to any abstract Gauss diagram. In Section 3.2 we give the conditions for an abstract Gauss diagram to represent a closed 3-manifold or a component of a knot in some closed orientable 3-manifold.

3.1. **Gauss diagrams for 3-manifolds with boundary.** In order to generalize the definition of Heegaard splitting to 3-manifolds with boundary, one replaces the handlebodies with the compression bodies. See, e.g. [Sc]. A *compression body* H is a 3-manifold with boundary obtained in a following way: let Σ be a closed surface. Take a cylinder $\Sigma\times[-1,1]$, attach to $\Sigma\times\{-1\}$ a collection of 2-handles. If the resulting manifold has any spherical components of the boundary, fill them with 3-balls. The surface $\Sigma\times\{1\}$ is called the outer boundary $\partial^+ H$ of the compression body, and the surface $\partial^- H = \partial H - \partial^+ H$ is called the inner boundary of H. If a boundary has one component only, namely its outer boundary $\partial^+ H$, we get a handlebody. The boundary of a compression body may consist of more than two components. The genus of the outer boundary is the genus of a compression body. Unlike the case of a handlebody, the genus does not determine the manifold completely. Two compression bodies of the same genus have homeomorphic outer boundaries, but there are no restrictions on the inner boundary, and the manifolds can be completely different. A Heegaard splitting of a manifold with boundary is then a union of two compression bodies $H^\pm$ glued using a homeomorphism of the outer boundaries $\partial^+ H^\pm$ of the compression bodies. The surface $\partial^+ H^+ = \partial^+ H^-$ is then called a splitting surface of the Heegaard splitting. We also allow now the families $\mathcal{M}^\pm$ to have different number of circles. The number of circles in each family $\mathcal{M}^\pm$ will be denoted by $g^\pm$.

Assume that G^d is an abstract decorated Gauss diagram with g circles in each family. Then one can look at the Gauss diagram as representing curves on $S(G^d)$ to which the discs are glued. Being more precise, one takes a collar $S(G^d)\times[-1,1]$ and attaches 2-handles to it along the curves in $\mathcal{M}^+\times\{1\}$ and $\mathcal{M}^-\times\{-1\}$. If a 3-sphere appears in

the boundary, it is filled with a 3-ball. Then $S(G^d)\times\{0\}$ is a Heegaard splitting surface for a manifold, possibly with boundary.

If one adds to one of the families of a Heegaard diagram a curve contractible on the splitting surface, gluing disc along it produces a spheric boundary component. This sphere is filled with a ball and the Heegaard splitting does not change. This move is called a bubble-move or a *B-move*. In terms of Gauss diagrams this looks like the addition to one of the families $\mathcal{M}^\pm$ a circle (called a *bubble*) without chords. This circle produces two cycles, one colored in a new color and the other in one of the existing colors. The choice of the existing color is the choice of the component of $S(G^d)-\mathcal{M}^\pm$ to which the bubble is added. The opposite move to this one, i.e. a removing of such a circle, is called B^{-1}-move.

Definition 3.1. *Two abstract Gauss diagrams are called B-*equivalent *if one can be obtained from the other by a sequence of B and B^{-1}-moves.*

The change on the Gauss diagram produced by a B-move is less significant than that made by an ε-move. Moreover, a B-move allows us to get rid of ε-move.

Proposition 3.2. *The ε-equivalence relation is generated by the HS and B equivalence relations.*

Proof. Let G_1^d, G_2^d be two abstract decorated Gauss diagrams. Assume that G_2^d can be obtained from G_1^d by an ε-move. We would like to show that G_2^d can be obtained from G_1^d by a sequence of B, B^{-1} and H, H^{-1}-moves. The diagram G_1^d differs from G_2^d by the signs of the chords joining one circle, say μ^+ in $\mathcal{M}_1^+$. Choose an edge in μ^+. Let (c,d) be its coloring. Perform a B-move adding a circle β to $\mathcal{M}_1^+$, colored by (e,c), where e is a new color. Perform an H-move, sliding β along μ^+. The new circle will have the direction opposite to that of μ^+, and the chords will have opposite signs. Perform an H^{-1}-move on μ^+ changing it into bubble. Remove this bubble with B^{-1}-move. The resulting diagram is G_2^d. See Figure 3.17. □

Lemma 3.3. *B-moves do not change the genus of $S(G^d)$.*

Proof. Let G'^d denote the diagram resulting from a Gauss diagram G^d after the B move. A B-move adds a circle to one of the families, without any chords joining it. Two new cycles and one new color are added, increasing the excess of coloring Δ_c by 1 and $|c|$ by 2. Hence $g_{S(G'^d)} = 1+\Delta_c+1+\frac{1}{2}(|h|-|c|-2) = g_{S(G^d)}$. □

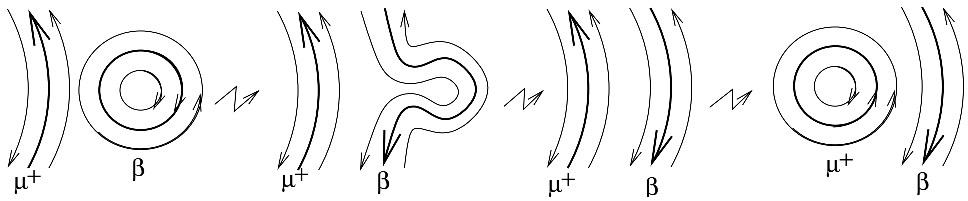


FIGURE 3.17. ε-move as bubble moves and handle slides

3.2. **The boundary and its genus.** We would like to understand which conditions a Gauss diagram should satisfy so that a manifold associated to it is closed. If the number of circles in each of $\mathcal{M}^\pm$ equals $g_{S(G^d)}$, we have an honest Heegaard splitting of a closed orientable 3-manifold, provided that neither $\mathcal{M}^+$ nor $\mathcal{M}^-$ separate the surface. See, e.g., [FM].

For a decorated Gauss diagram G^d consider a pair of graphs $C^\pm = C^\pm(G^d)$. The vertices of both graphs are the colors used in G^d. Two vertices p, q of, say, C^+ have a common edge if there is an edge or a circle without chords in $\mathcal{M}^-$ with a coloring (p, q) or (q, p). The following Lemma is obvious:

Lemma 3.4. *The connected components of $C^\pm$ correspond to the connected components of $S(G^d) - \mathcal{M}^\pm$ respectively.*

Given a Gauss diagram G^d, we will denote by $\partial g^\pm$ the sum of the genera of the components of $\partial^- H^\pm$ respectively.

Proposition 3.5. *Let G^d be a Gauss diagram, and let $k^\pm$ be the number of components in $C^\pm(G^d)$. Then $\partial g^\pm = k^\pm - g^\pm + g_{S(G^d)} - 1$.*

Proof. Let $S^\pm$ be surfaces obtained from $S(G^d)$ by a surgery on the curves of $\mathcal{M}^\pm$ respectively. Each $S^\pm$ is the inner boundary of the compression body $H^\pm$ together with several spheres that will be capped. Then the Euler characteristic of, say, S^+ is:

$$\begin{aligned}\chi(S^+)) &= \chi((S(G^d) - \mathcal{M}^+) \cup 2g^+ \text{ discs}) = \\ &= \chi(S(G^d)) + 2g^+ = 2 - 2g_{S(G^d)} + 2g^+\end{aligned}$$

This is also the sum of the Euler characteristics of the components S_j of S^+, each of which has genus g_j. Their number is exactly the number of the components of $S(G^d) - \mathcal{M}^+$. By Lemma 3.4 this number is k^+. Consequently,

$$2 - 2g_{S(G^d)} + 2g^+ = \sum_{j=1}^{k^+} (2 - 2g_j) = 2k^+ - 2\sum_{j=1}^{k^+} g_j = 2k^+ - 2\partial g^+,$$

and the assertion follows. $\square$

Corollary 3.6. *If $\partial g^{\pm} = 0$, then the manifold associated to G^d is a closed orientable 3-manifold. If $\partial g^+ = 0, \partial g^- = 1$, or vice versa, then the manifold associated to G^d is a complement of a knot in some closed orientable 3-manifold.*

Proof. Let $H^{\pm}$ be compression bodies obtained from $S(G^d) \times [-1, 1]$ by gluing 2-handles along the curves of $\mathcal{M}^{\pm}$ respectively and filling in 3-balls. For a handlebody we need $\partial g^{\pm} = 0$, since $\partial^- H^{\pm}$ has to be empty. To get a single torus as the inner boundary of H^- and a single boundary component of M, we need $\partial g^+ = 0, \partial g^- = 1$. $\square$

Definition 3.7. *Two abstract Gauss diagrams will be called* equivalent *if one can be obtained from the other by a sequence of R, H, S, B-moves and their opposites.*

Proposition 3.8. *The numbers $\partial g^{\pm}$ are invariants of the equivalence classes of abstract Gauss diagrams.*

Proof. Consider a B-move. By Lemma 3.3, it preserves $g_{S(G^d)}$. When a bubble is added to $\mathcal{M}^+$, a vertex is added to both graphs $C^{\pm}(G^d)$. In C^- this vertex is joined to another by an edge, and k^- does not change, but in C^+ no edge joins it. Hence k^+ increases by 1. The number of circles also increases by 1 only in $\mathcal{M}^+$. Computing $\partial g^{\pm}$ by formula of Proposition 3.5 shows that the values are preserved by B-move.

Consider an S-move. Both $g^{\pm}$ increase by 1. No new colors are added, and the edges added are colored by (c, c). Hence no change is made on $k^{\pm}$. From Lemma 2.13 we know that $g_{S(G^d)}$ increases by 1. Hence $\partial g^{\pm}$ are preserved by S-move.

Consider an H-move. This time $g_{S(G^d)}$ is preserved (Lemma 2.12), and $g^{\pm}$ too. According to Lemma 3.4, $k^{\pm}$ is the number of connected components of $S(G^d) - \mathcal{M}^{\pm}$. Assume, for convenience, that the slide occurs in the $\mathcal{M}^+$ family. We would like to understand how it affects the connected components of $S(G^d) - \mathcal{M}^{\pm}$. Since only the curves of $\mathcal{M}^+$ are affected by the move, there is no change on the connected components of $S(G^d) - \mathcal{M}^-$, and k^- is preserved.

Let μ_1 slide along μ_2. Both of them are boundary components of some Σ', which is a connected component of $S(G^d) - \mathcal{M}^+$. A handle slide looks on Σ' like replacing two components of $\partial\Sigma'$ by their connected sum. Assume that Σ' is glued along μ_1 to another component Σ'' of $S(G^d) - \mathcal{M}^+$. A slide of μ_1 along μ_2 looks like the addition to Σ'' of an annulus on a thin band. No new components are created in $S(G^d) - \mathcal{M}^+$, and no components are glued together. See Figure 3.18.

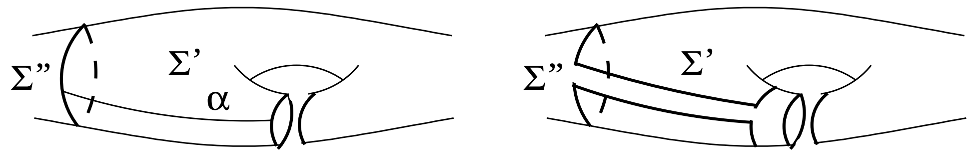


FIGURE 3.18. Handle slide on the splitting surface

If both copies of μ_1 are boundary components of Σ', then an annulus on a thin band is glued to the same Σ'. There is no change in the other connected components of $S(G^d) - \mathcal{M}^+$. Hence k^+ is preserved in both cases. We conclude that $\partial g^\pm$ is invariant under H-move.

Consider an R-move. There is no change in $g^\pm$. If the edges, where the intersections appear, belonged to different cycles, then the number of colors increases by one, and the number of cycles does not change. This means that the genus of the surface associated to the changed diagram G'^d is $g_{S(G'^d)} = 1 + \Delta_c - 1 - \frac{1}{2}(|h| + 2 - |c|) = g_{S(G^d)}$. In this case the only new color, in both $C^\pm$, is a vertex of valency 1, joining the existing components. See also Figure 2.7. There is no other change in the graphs $C^\pm$. If the edges where the intersections appeared belonged to the same cycle, then the number of colors increases by 2. The number of cycles $|c|$ also increases by 2. This means that $g_{S(G'^d)} = 1 + \Delta_c - \frac{1}{2}(|h| + 2 - |c| - 2) = g_{S(G^d)}$. One of the new colors, in both $C^\pm$, is a vertex of valency 1, joining the existing components, as in the previous case. There are two more new colors replacing one old, and this looks like splitting the vertex. Just as in the case of handle slide, these vertices belong to the same component, since in both graphs a path can be found between them. We conclude that $k^\pm$ is also preserved and $\partial g^\pm$ is invariant under R-move. □

Theorem 3.9. *Closed connected orientable manifolds are in one-to-one correspondence with the equivalence classes of abstract decorated Gauss diagrams satisfying* $\partial g^\pm = 0$.

Proof. Obviously, two equivalent diagrams have the same associated manifold. Now assume that we have a Gauss diagram G_1^d, satisfying $\partial g^\pm = 0$, that is equivalent to a Gauss diagram G_2^d with $k^\pm = 1$. By Proposition 3.8, $g^\pm = g_{S(G_2^d)}$. By Corollary 3.6, the manifold associated to G_2^d is closed. By Theorem 2.16 we are done. It remains to show that any abstract Gauss diagram G_1^d satisfying $\partial g^\pm = 0$ is equivalent to a Gauss diagram G_2^d with $k^\pm = 1$.

We can assume that $k^+ > 1$. Let Σ' be a connected component of $S(G_1^d) - \mathcal{M}^+$. If Σ' is not a disc with holes, then $\Sigma' \times \{-1\}$ together

with discs glued to its boundary components is a non-trivial component of $\partial^- H^+$. This contradicts the assumption $\partial g^+ = 0$. Thus Σ' is a disc with holes.

Since $k^+ > 1$, there is a circle μ^+ in $\partial\Sigma'$, that is not glued to any other component of $\partial\Sigma'$. Isotope μ^+, probably performing some R-moves, so that μ^+ can be connected by arcs disjoint from $\mathcal{M}^-$ to all other components of $\partial\Sigma'$. Then μ^+ can slide along all boundary components of Σ'. A handle slide looks on Σ' like replacing two components of $\partial\Sigma'$ by their connected sum, hence the number of components of $\partial\Sigma'$ decreases. From the proof of Proposition 3.8 we know that handle slides do not change $k^\pm$.

The handle slides of μ^+ along the components of $\partial\Sigma'$ change Σ' into a disc. Since $\mathcal{M}^- \cap \Sigma'$ consists of disjoint arcs and bubbles, all these can be removed by B^{-1} and R^{-1}-moves. This will change μ^+ into a bubble. Remove the bubble by another B^{-1}-move. This will decrease k^+ by 1. Thus G_1^d is equivalent to a diagram with $k^+ = 1$. □

3.3. **The fundamental group and the first homology group.** Since we reconstruct M from its Gauss diagram as a cell complex, there is an obvious way to obtain its fundamental group. A Gauss diagram is a graph, which we denote also by G for convenience.

Proposition 3.10. *Let G be an abstract Gauss diagram with all cycles colored differently and M the associated manifold. Let T be the maximal tree subgraph of G, containing all chords. The generators of $\pi_1(M)$ are the edges of $G - T$, and the relations are defined by the cycles of G and the circles of both families.*

Proof. The fundamental group of the ribbon graph $\Gamma(G)$ coincides with the fundamental group of G, which is the free group generated by the edges of $G - T$. To construct the splitting surface one glues discs to the boundary components of $\Gamma(G)$, which means to the cycles. For any cycle write a relation going along it and writing down only the edges in $G - T$. In the cycles part of the edges are traversed in the direction opposite to their orientation. In this case write the inverse of the generator.

Gluing 2-handles along the circles of G means introducing another set of relations. The circles are also sequences of edges, and only those in $G - T$ should be written. Filling in 3-balls does not affect the fundamental group. □

Remark: There is another, simpler way to obtain the fundamental group for closed orientable manifolds, if the number of circles in each family of the Gauss diagram equals the genus of the associated surface

and no separation occurs. This is the method of Proposition 2.17.

The first homology group of M is the abelian group generated by the edges of $G-T$. The relations coming from the circles in the Gauss diagram remain unchanged. In some cycles there could be edges appearing twice, with opposite signs. Hence in $H_1(M)$ in the relations coming from the cycles one writes only those generators that are colored by two different cycles.

3.4. **Examples.** In both following examples we represent the diagrams with all cycles colored differently, hence the decorations are omitted.

Thickened torus: $T^2 \times I$ can be represented by a diagram in Figure 3.19, which also shows how the curves are situated on the surface. The count of cycles gives $|c| = 8$, so $g_{S(G)} = 1$, which equals the number of curves in each family. One can also check, using the graphs $C^{\pm}$ of Section 3.2, that each curve separates the surface into two components. The Proposition 3.5 then implies that both boundary components are tori.

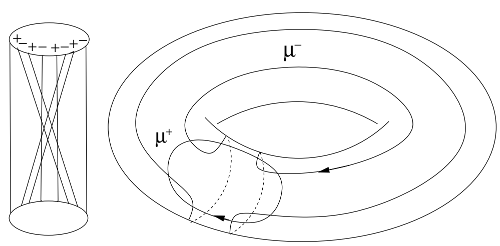


FIGURE 3.19. $T^2 \times I$

Unknot complement: The diagram of Figure 3.20 has 4 cycles, which implies genus $g = 1 = g_{S(G)}$, and drawing the graphs $C^{\pm}$ one can see that the top family does not separate the surface, while the bottom does, and we have two components. Let us count the fundamental group. The maximal tree T is thickened in the diagram. The generators are the arcs in $G-T$, as shown in Figure 3.20. The fundamental group then is:

$$\langle e_1, e_2, e_3, g_1, g_2 | e_1e_2e_3, g_1g_2, e_1g_1^{-1}e_1^{-1}g_2^{-1}, e_2g_1, e_2^{-1}e_3g_2, e_3^{-1}\rangle,$$

where the last four relations come from the cycles. This gives us $\pi_1(M) = Z$. The manifold is the complement of the unknot in S^3.

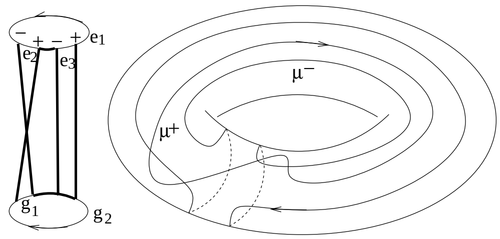


FIGURE 3.20. The complement of unknot in S^3

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